Classical and Refined Beam and Plate Theories: A Brief Technical Review

Rajesh A Shetty[1*], Dushyanthkumar G L[1] and Deepak S A[2]

[1] Mechanical Engineering Dept., Sahyadri College of Engineering & Management, Adyar, Mangaluru-575007
[2]School of Mechanical Engineering, Reva University, Bengaluru, Karnataka - 560064
*Email: rajeshshetty.mech@sahyadri.edu.in

ABSTRACT

In this paper, a brief review of classical and refined beam and plate theories has been presented. For easy understanding of the concept of beam and plate theories, the discussion has been started with an introduction to the simplest, yet extensively used classical beam and plate theories. Later, the discussion has been extended to the refined or shear deformation beam and plate theories which are considered to be more accurate and analytically more acceptable compared to the classical theories. The equations governing the elementary or classical theories are highly simple, hence are could be discussed in undergraduate and postgraduate courses. Whereas, the displacement functions and governing differential equations connected with refined or shear deformation theory formulation are considerably complex in comparison to the classical theory formulation. The inclination towards refined beam and plate theories is developed due to the difficulty in formulation of thick beams and plates using classical theories. The classical theories could be used only for the investigation of slender beams and thin plates as these theories can capture only the bending deflections. However, in case of thick beams and plates, the contributions from both bending and shear deflections need to be taken into account in the theory formulation. This could be achieved by using refined beam and plate theories. In this regard, a brief review of various well-known refined beam and plate theories has been presented in this paper. Also, a brief discussion on pros and cons of various theories has been presented.

***Keywords:* Beam and Plate Theories, Classical Theories, Refined Theories, Shear Deformation Theories, Bending Deflection, Shear Deflection.**

NOMENCLATURE

A	-	Beam cross-sectional area
b	-	Beam width
D	-	Plate flexural rigidity
E	-	Modulus of elasticity
G	-	Shear modulus
h	-	Beam height or plate thickness
I	-	Moment of inertia
k	-	Shear coefficient
t	-	Time variable
u, v	-	In-plane displacements in x & y-directions
u_0, v_0	-	Axial displacements in x & y-directions
w	-	Lateral displacement in z-direction
w_b	-	Bending component of lateral deflection w
w_s	-	Shear component of lateral deflection w
w_0	-	Lateral displacement about mid-plane
x, y, z	-	Cartesian coordinates
q	-	Intensity of transverse distributed load
ϕ_x	-	Rotation of the normal about y-axis
ϕ_y	-	Rotation of the normal about y-axis
ρ	-	Density of the material

1. INTRODUCTION

Beam Theories: Beams are the structural elements which are primarily designed for withstanding applied transverse loads by developing bending stresses. In practice, beams are the 3-dimensional elements. However, for the analytical modeling purpose, the beams can be considered as one-dimensional elements by taking into consideration of the large longitudinal dimension in comparison to the small cross-sectional dimensions. Most of the beam theory equations are derived by considering beams as one-dimensional elements. The theoretical study of beams using beam theories is an approximate, but easy and effective way of investigation of behavior of beams subjected to transverse loads.

A vast literature is available with regard to the static, vibration and buckling analysis of beams using beam theories. Some of the important theories proposed by researchers for the study of beams are, Euler-Bernoulli Beam Theory (EBT) [1], Timoshenko Beam Theory (TBT) [2], Levinson Beam Theory (LBT [3]), Reddy Beam Theory (RBT) [4] and so on. Euler-

Bernoulli Beam Theory, proposed in 18th century, is a basic or elementary theory in the domain of beam theories. Timoshenko Beam Theory, proposed in 1921, is a first-order shear deformation beam theory. Levinson Beam Theory (1981) and Reddy Beam Theory (1984) are the higher-order shear deformation beam theories. The first-order and higher-order shear deformation beam theories are the refined version of Euler-Bernoulli Beam Theory, which are developed by researchers to overcome the drawbacks posed by the elementary beam theory. A large number of research articles are available on beam analysis using Euler-Bernoulli Beam Theory and other just mentioned first-order and higher-order shear deformation beam theories.

Plate Theories: Analogous to beams, plates are also the structural elements which are designed for resisting applied transverse loads by developing bending stresses. The main difference between the beam and plate elements is that, for theoretical study purpose, the beams are considered as one-dimensional elements, whereas, plates are treated as two-dimensional elements. This simplification is possible as, in case of plates the in-plane dimensions are large compared to plate thickness. Hence, while formulating a plate theory, this factor could be considered and the plate equations could be derived by simplifying plate as a two-dimensional element. The theoretical study of plates using plate theories is an approximate way of investigating the static and dynamic behavior of plates.

Classical Plate Theory (CPT) [5], proposed in 19th century, is the basic or elementary theory in the domain of plate theories. Classical Plate Theory is also known as "Kirchhoff-Love Theory." The refined plate theories or shear deformations plate theories available in the domain of plate theories are, Reissner Plate Theory (1945) [6], Mindlin Plate Theory (1951) [7], Levinson Plate Theory (1980) [8], Reddy Plate Theory (1984) [9], Refined Plate Theory (2001) [10], New First-order Shear Deformation Plate (2007) [11] and so on. Reissner Plate Theory is a stress based theory. The formulation of the theory begins with assumptions of the stress functions. Mindlin Plate Theory and New First-order Shear deformation plate theory are the displacement based first-order shear deformation plate theories. Levinson Plate Theory, Reddy Plate Theory and Refined Plate Theory are the higher-order shear deformation plate theories. The refined plate theories could be used for the plate analysis whenever the results predicted by Classical Plate theory are unsatisfactory. In literature, a vast research articles are available pertaining to the analytical study of static, vibration and buckling behavior of plates using classical and refined plate theories.

In this article, a brief review of various important beam and plate theories has been presented. To begin with, a discussion on classical beam and plate theories has been presented. The drawbacks of Euler-Bernoulli Beam Theory and Classical Plate Theory have been discussed in detail. Also, the importance of shear deformation theories over the classical theories has been highlighted. Further, a short discussion on formulation part of various beam and plate theories such as displacement functions and governing differential equations has been presented.

2. REVIEW OF BEAM THEORIES

2.1 Euler-Bernoulli Beam Theory

Euler-Bernoulli Beam Theory (EBT), also known as "Elementary Beam Theory" is the first beam theory formulated for the analysis of beams. It was first proposed in 18th century. The expressions for displacements of a transversely loaded beam given by EBT are as follows [12, 13]:

$$u = -z\frac{dw_0}{dx} \quad (1) \qquad v = -z\frac{dw_0}{dy} \quad (2) \qquad w = w_0(x) \quad (3)$$

The governing differential equation used for obtaining the transverse deflection of a beam subjected to static bending given by EBT is

$$EI\frac{d^4 w_0}{dx^4} = q(x) \tag{4}$$

The governing equation is a fourth-order differential equation with w_0 as an unknown. By integrating Eq. (4) and applying the beam end conditions, the value of transverse deflection w_0 can be determined. In literature, it has been discussed comprehensively that the deflections predicted by EBT are only accurate, if the beams under consideration are slender beams. In case of short or thick beams, EBT underestimate the deflection values. EBT can only capture the bending deflection. However, the contribution of shear deflection cannot be ignored in case of short or thick beams.

Further, the governing equation for transverse vibrations of a beam given by EBT is

$$EI\frac{\partial^4 w_0}{\partial x^4} + \rho A\frac{\partial^2 w_0}{\partial t^2} = q(x,t) \tag{5}$$

By considering the effects of rotary inertia, the Eq. (5) can be written as follows:

$$EI\frac{\partial^4 w_0}{\partial x^4} + \rho A\frac{\partial^2 w_0}{\partial t^2} + \rho I\frac{\partial^4 w_0}{\partial x^2 \partial t^2} = q(x,t) \tag{6}$$

As discussed earlier, EBT is prone to underestimate the deflections in case of thick beams. This means, EBT assumes beams are stiffer. Hence, the frequencies predicted by EBT are higher. Also, the buckling loads predicted by EBT are also higher. The drawbacks associated with EBT resulted in the development of refined beam theories which could formulate a beam by considering the effects of shear deformation also.

2.2 First-order Shear Deformation Beam Theories

To overcome the drawbacks posed by elementary beam theory, a new class beam theories known as, First-order shear deformation beam theories are came into existence. These theories can consider the effects of shear deformation also in the beam formulation in addition to the bending deformation. The primary drawback linked with first-order theories is the difficulty in deciding the correct shear coefficients. The formulation of first-order theories is such that, the theories cannot satisfy the shear stress free surface conditions at the top and bottom surfaces of the beams. To account for this it is necessary to use suitable shear coefficients to slightly improve the results yielded by TBT in comparison to exact theory and other shear deformation beam theories.

2.2.1 Timoshenko Beam Theory

Timoshenko Beam Theory (TBT), proposed in 1921, is a highly popular and extensively used first-order shear deformation beam theory. The expressions for displacement of a beam given by TBT are [12],

$$u = -z\phi_x \quad (7) \qquad w = w_o(x, y) \quad (8)$$

The above displacement expressions contain two unknown functions. According to TBT, the differential equations governing the beam behavior are given by

$$\rho A \frac{\partial^2 w_0}{\partial t^2} - kGA\left(\frac{\partial^2 w_0}{\partial x^2} - \phi_x\right) - q(x,t) = 0 \tag{9}$$

$$\rho I \frac{\partial^2 \phi}{\partial t^2} - EI\frac{\partial^2 \phi}{\partial x^2} - kGA\left(\frac{\partial w_0}{\partial x} - \phi\right) = 0 \tag{10}$$

In the above equations, k is the shear coefficient. The value of shear coefficient k is reported in literature for various cross-sections of the beams. The above differential equations need to be solved to obtain two unknown variables. The lateral displacement predicted by TBT involves both the bending deflection as well as the shear deflection. However, the use of shear coefficient is the main drawback linked with the TBT. Also, TBT predicts constant shear stress across the thickness of the beam. Therefore, TBT cannot satisfy the shear stress free surface conditions. These just discussed some of the drawbacks linked with first-order theories led to the development of a new class of theories known as "Higher-order shear deformation theories."

2.3 Higher-order Shear Deformation Beam Theories

To deal with ambiguity in deciding suitable shear coefficients in case of first-order shear deformation theories, the interest is later switched over to higher-order shear deformation theories. In case of higher-order shear deformation theories, the displacement field of the theories can predict realistic transverse shear stress distribution through the beam height. Also, the displacement field can satisfy zero shear stress surface conditions. Hence, the need for using shear coefficients in the beam formulation can be avoided.

2.3.1 Levinson Beam Theory

Levinson Beam Theory (LBT), proposed in 1981, is a higher-order shear deformation beam theory. This beam theory does not use a shear coefficient as the displacement field of the theory can predict realistic shear stress distribution across the beam thickness. The displacement field is [3],

$$u = u_0 + z\left[\phi_x - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\phi_x + \frac{\partial w_0}{\partial x}\right)\right] \tag{11}$$

$$w = w_0(x, y) \tag{12}$$

(12)

The displacement filed of the theory involves two unknown functions. The derived differential equations of the Levinson Beam Theory are given by

$$\frac{2}{3}\frac{\partial}{\partial x}\left[AG\left(\phi + \frac{\partial w_0}{\partial x}\right)\right] = \rho A\frac{\partial^2 w_0}{\partial t^2} - q(x,y,t) \tag{13}$$

$$\frac{2}{3}AG\left(\phi + \frac{\partial w_0}{\partial x}\right) + \frac{1}{5}\frac{\partial}{\partial x}\left[EI\left(\frac{\partial^2 w_0}{\partial x^2} - 4\frac{\partial \phi}{\partial x}\right)\right] = \frac{\rho I}{5}\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial x} - 4\phi\right) \tag{14}$$

The above two coupled differential equations need to be solved to determine the two unknown variables. The beam deflections and frequencies predicted by LBT are quite accurate. Unlike Timoshenko Beam Theory, LBT does not require a shear coefficient. The derivations of this theory are easy to understand as the differential equations of LBT are derived by Newtonian approach. Further, it is easy to understand the physical meaning of the boundary conditions used in case of beam analysis using LBT.

2.3.2 Reddy Beam Theory

Reddy Beam Theory (RBT), proposed in 1984, is a higher-order shear deformation beam theory. This theory is classified under the category of third-order theories as the displacement field of this theory involves third order terms. The theory can predict realistic shear stress distribution across the beam thickness. The displacement field of the theory is [4],

$$u = u_0 + z\left[\phi_x - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\phi_x + \frac{\partial w_0}{\partial x}\right)\right] \tag{15}$$

$$w = w_0(x, y) \tag{16}$$

The displacement field of Reddy's Beam Theory is same as that of LBT. But, the difference between these two theories lies in the approach used in deriving the governing differential equations. In case of RBT, the governing equations are derived by energy principles, whereas, in case of LBT the differential equations are derived by Newtonian approach. The governing equations of RBT are given as:

$$AE\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + p = \rho A\frac{\partial^2 u}{\partial t^2} \tag{17}$$

$$\begin{aligned}&\frac{\partial}{\partial x}\left\{AE\frac{\partial w}{\partial x}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + \frac{8}{15}GA\left(\phi + \frac{\partial w}{\partial x}\right)\right\}\\ &-\frac{\partial^2}{\partial x^2}\left[\frac{1}{21}EI\frac{\partial^2 w}{\partial x^2} - \frac{16}{105}EI\frac{\partial \phi}{\partial x}\right] + q\\ &= \rho A\frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left[\frac{1}{21}\rho I\frac{\partial^3 w}{\partial t^2 \partial x} - \frac{16}{105}\rho I\frac{\partial^2 \phi}{\partial t^2}\right]\end{aligned} \tag{18}$$

$$-\frac{\partial}{\partial x}\left[\frac{68}{105}EI\frac{\partial \phi}{\partial x} - \frac{16}{105}EI\frac{\partial^2 w}{\partial x^2}\right] + \frac{8}{15}GA\left(\phi + \frac{\partial \phi}{\partial x}\right) = -\frac{68}{105}EI\frac{\partial^2 \phi}{\partial t^2} + \frac{16}{105}\rho I\frac{\partial^2 w}{\partial x \partial t^2} \tag{19}$$

There are three coupled differential equations. The number of variables can be reduced to two by ignoring the variable corresponding to the axial displacement. It is to be observed that, the equations governing the RBT are more complex compared to those of LBT, even though both the theories are formulated based upon the same displacement field. RBT is considered to be more accurate compared to LBT as the equations of RBT are derived through variable consistent approach.

2.3.3 Single Variable Refined Beam Theory

This Single Variable Refined Beam Theory (SVRBT) [14] involves only one unknown function. This is also a higher-order shear deformation beam theory. In the expressions for displacements, we have third order terms. No need of using a shear coefficient. The displacement field is

$$u = -z\frac{\partial w_b}{\partial x} + \frac{2(1+\mu)}{Eb}\left[\frac{3}{10}\left(\frac{z}{h}\right) - 2\left(\frac{z}{h}\right)^3\right] \times \left[-EI\frac{\partial^3 w_b}{\partial t^2} + \rho I\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_b}{\partial x}\right)\right] \quad (20)$$

$$w = w_b + \frac{12(1+\mu)}{5Ebh}\left[-EI\frac{\partial^2 w_b}{\partial x^2} + \rho I\frac{\partial^2 w_b}{\partial t^2}\right] \quad (21)$$

Even though theory is a higher-order theory, the formulation of the theory involves only one unknown function. Most of the equations of this theory have strong similarity to the equations of Euler-Bernoulli Beam Theory (EBT). Therefore, the beam analysis using this theory involves slightly higher effort required in case of beam analysis using EBT.

The governing equation of this theory is derived by using the gross equilibrium equations. The differential equation governing the beam behavior is given by

$$EI\frac{d^4 w_b}{dx^4} = q(x) \quad (22)$$

The above differential equation is similar to the governing equation of EBT. By solving the above differential equation, we can obtain the value of bending deflection (i.e. w_b). Next, the total lateral deflection 'w' can be calculated by using the following equation:

$$w = w_b - \frac{h^2(1+\mu)}{5}\frac{d^2 w_b}{dx^2} \quad (23)$$

The analysis of beams using SVRBT is similar to that of EBT as most of the equations of this theory are as simpler as those of EBT.

For the case of vibration analysis, the derived differential equation of SVRBT can be written as follows:

$$EI\frac{\partial^4 w_b}{\partial x^4} - \rho I\left[1 + \frac{12(1+\mu)}{5}\right]\frac{\partial^4 w_b}{\partial x^2 \partial t^2} + \rho A\frac{\partial^2 w_b}{\partial t^2} + \frac{\rho^2 I}{E}\frac{12(1+\mu)}{5}\frac{\partial^4 w_b}{\partial t^4} = q(x,t) \quad (24)$$

In the above equation we have only one unknown variable. By solving the above equation for unknown variable (w_b), we can study the vibration behavior of a beam under consideration. The above differential equation includes both the effects of shear deformation as well as rotary inertia.

3. REVIEW OF PLATE THEORIES

3.1 CLASSICAL PLATE THEORY

Classical Plate Theory (CPT), proposed in 19th century, is a basic and easy to use plate theory available in the literature plate theories. This theory is also known as "Kirchhoff-Love Theory." The formulation of the theory along with properly prescribed boundary conditions was ready by approximately around 1850. Even though the governing equation of the theory was available since 1815, the theory could not be used to its full potential till 1850, due to the difficulty in understanding of the free edge boundary conditions associated with the theory. The convincing explanation about boundary conditions is obtained after Kirchhoff used energy principles to derive the governing equations and boundary conditions of CPT. The displacement field of CPT is given by [12, 13],

$$u = -z\frac{\partial w_0}{\partial x} \quad (25) \qquad v = -z\frac{\partial w_0}{\partial y} \quad (26)$$

$$w = w_0(x,y) \quad (27)$$

The displacement field of the theory involves only one variable (w_0). Using the above expressions for displacements, we can write the expressions for strains and stresses of a given plate. Next, the governing differential equation of CPT can be written as follows:

$$D\nabla^2\nabla^2 w_0 = q(x,y) \quad (28)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

The above differential equation is for the case of static analysis of a plate. By solving the above equation, one gets the solution for static deflection of a plate. Next, the governing equation for the case of vibration of a plate can be written as:

$$D\nabla^2\nabla^2 w_0 + \rho h\frac{\partial^2 w_0}{\partial t^2} = q(x,y,t) \quad (29)$$

In the above equation, the effects of shear deformation and rotary inertia are ignored. The use of above equation for the case of thick plates would result in the overestimation of frequencies. To overcome this drawback, one can use the shear deformation plate theories.

3.2 First-order Shear Deformation Plate Theories

3.2.1 Mindlin Plate Theory

Mindlin Plate Theory (MPT), proposed in 1921, is a first-order shear deformation plate theory. The formulation of this theory is analogous to that of Timoshenko Beam Theory (TBT) proposed in 1921. Difference is that TBT is one-dimensional theory, whereas, MPT is two-dimensional theory. As discussed in case of beams, first-order plate theories predict constant shear stress across the plate thickness. Therefore, one will have non-zero shear stresses at the top and bottom surfaces of the plate. This theory uses a shear correction factor. The displacement field of the MPT can be written as [12, 13]:

$$u = -z\phi_x \quad (30) \qquad v = -z\phi_y \quad (31) \qquad w = w_o(x, y) \quad (32)$$

There are three unknown variables are involved in the displacement field of MPT. For the complete plate analysis using MPT, one needs to solve the MPT equations to determine these three variables. The differential equations connected with MPT for the case of vibrations of a thick plate can be written as:

$$D\left(\frac{\partial^2\phi_x}{\partial x^2} + \frac{1-\mu}{2}\frac{\partial^2\phi_x}{\partial y^2} + \frac{1+\mu}{2}\frac{\partial^2\phi_y}{\partial x\partial y}\right) + kGh\left(\frac{\partial w_0}{\partial x} - \phi_x\right) - \frac{\rho h^3}{12}\frac{\partial^2\phi_x}{\partial t^2} = 0 \quad (33)$$

$$D\left(\frac{\partial^2\phi_y}{\partial y^2} + \frac{1-\mu}{2}\frac{\partial^2\phi_y}{\partial x^2} + \frac{1+\mu}{2}\frac{\partial^2\phi_x}{\partial x\partial y}\right) + kGh\left(\frac{\partial w_0}{\partial y} - \phi_y\right) - \frac{\rho h^3}{12}\frac{\partial^2\phi_y}{\partial t^2} = 0 \quad (34)$$

$$-kGh\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2} - \frac{\partial \phi_x}{\partial x} - \frac{\partial \phi_y}{\partial y}\right) + \rho h\frac{\partial^2 w_0}{\partial t^2} = q(x, y, t) \quad (35)$$

The above equations are coupled differential equations. By solving above three equations, we can have the complete solution of a given plate. In above equations, the notation '*k*' denotes the shear coefficient. Based upon the literature, one can decide the value of shear coefficient. The shear coefficient is needed to approximately map the results of Mindlin Plate Theory with those of exact theory and other shear deformation plate theories.

3.3 Higher-order Shear Deformation Plate Theories

3.3.1 Levinson Plate Theory

Levinson Plate Theory (LPT), proposed in 1980, is a higher-order shear deformation theory. This theory is a third-order plate theory. In case of LPT, no shear coefficient is required. The displacement field of this theory involves three unknown variables. The displacement field of LPT is [8],

$$u = u_0 + z\left[\phi_x - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\phi_x + \frac{\partial w_0}{\partial x}\right)\right] \quad (36)$$

$$v = v_0 + z\left[\phi_y - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\phi_y + \frac{\partial w_0}{\partial y}\right)\right] \quad (37)$$

$$w = w_0(x, y) \quad (38)$$

Using the above displacement expressions, one can obtain the strains and stresses of a given plate under consideration. The differential equations related to Levinson Plate Theory can be written as follows:

$$\frac{2}{3}Gh\left(\nabla^2 w_0 + \psi\right) + q(x, y, t) = \rho h\frac{\partial^2 w_0}{\partial t^2} \quad (39)$$

$$\frac{2}{5}D\left[(1-\mu)\nabla^2\phi_x + (1+\mu)\frac{\partial\psi}{\partial x} - \frac{1}{2}\frac{\partial}{\partial x}\left(\nabla^2 w_0\right)\right] - \frac{2}{3}Gh\left(\phi_x + \frac{\partial w_0}{\partial x}\right) = \frac{\rho h^3}{60}\frac{\partial^2}{\partial t^2}\left(4\phi_x - \frac{\partial w_0}{\partial x}\right) \quad (40)$$

$$\frac{2}{5}D\left[(1-\mu)\nabla^2\phi_y + (1+\mu)\frac{\partial\psi}{\partial y} - \frac{1}{2}\frac{\partial}{\partial y}\left(\nabla^2 w_0\right)\right] - \frac{2}{3}Gh\left(\phi_y + \frac{\partial w_0}{\partial y}\right) = \frac{\rho h^3}{60}\frac{\partial^2}{\partial t^2}\left(4\phi_y - \frac{\partial w_0}{\partial y}\right) \quad (41)$$

Similar to MPT, in case of Levinson Plate Theory also we have three coupled differential equations. The above differential equations have been derived by Newtonian approach.

3.3.2 Reddy Plate Theory

Reddy Plate Theory, proposed in 1984, is a higher-order shear deformation plate theory. Likewise Levinson Plate Theory, this theory also does not require a shear coefficient. The displacement field of the Reddy Plate Theory is same that of Levinson Plate Theory. However, Reddy's theory formulation differs from Levinson theory in the approach used in deriving the governing equations. Levinson has used Newtonian approach, whereas, Reddy has used energy principles to derive the governing differential equations. The displacement field is given by [9],

$$u = u_0 + z\left[\phi_x - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\phi_x + \frac{\partial w_0}{\partial x}\right)\right] \quad (42)$$

$$v = v_0 + z\left[\phi_y - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\phi_y + \frac{\partial w_0}{\partial y}\right)\right] \quad (43)$$

$$w = w_0(x, y) \quad (44)$$

The governing differential equations of Reddy's Plate Theory are not presented here as they are too lengthy. Those equations are available in the Ref. [].

3.3.3 Refined Plate Theory

Refined Plate Theory (RPT), proposed in 2001, is a two variable shear deformation plate theory. The analysis of plates using RPT involves two variables. The two variables involved are bending deflection (w_b) and shear deflection (w_s). The displacement field of RPT can be written as [10]:

$$u = -z\frac{\partial w_b}{\partial x} + h\left[\frac{1}{4}\frac{z}{h} - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\frac{\partial w_s}{\partial x} \quad (45)$$

$$v = -z\frac{\partial w_b}{\partial y} + h\left[\frac{1}{4}\frac{z}{h} - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\frac{\partial w_s}{\partial y} \quad (46)$$

$$w = w_b(x, y) + w_s(x, y) \quad (47)$$

The unknown variables in the above equations can be determined by solving the following two differential equations:

$$\nabla^2\nabla^2 w_b = \frac{q(x, y)}{D} \quad (48)$$

$$\frac{1}{84}\left(\nabla^2\nabla^2 w_s\right)-\frac{5(1-\mu)}{h^2}\nabla^2 w_s=\frac{q(x,y)}{D} \quad (49)$$

The above two differential equations are uncoupled and hence, these equations can be solved separately. Therefore, obtaining the solution using RPT is simpler when compared with other shear deformation plate theories. Most of the differential equations associated with shear deformation theories are coupled and hence, solution is harder.

Further, RPT displacement field predicts realistic quadratic shear stress distribution across the plate thickness. Therefore, no need to use a shear coefficient in case of RPT. Many equations of RPT are having strong similarity to the equations of CPT. Hence, RPT is easy to understand and RPT equations can be easily dealt with in the similar lines CPT equations.

3.3.4 Single Variable Refined Plate Theory

Single Variable Refined Plate Theory (SVRPT) [15] is a theory developed based upon two variable RPT discussed in the preceding section. The displacement field of the SVRPT involves only one variable. Also, the moment and shear force expressions of a plate given by SVRPT are similar to those of expressions given by CPT. One can easily understand SVRPT in the similar lines of CPT. The governing differential equation of SVRPT is given by

$$D\nabla^2\nabla^2 w_b = q(x,y) \quad (50)$$

The above differential equation is strikingly similar to the governing equation of CPT. Using the above equation one can [5]determine the bending deflection. The total plate deflection can be determined by using the following equation:

$$w = w_b - \frac{h^2}{5(1-\mu)}\nabla^2 w_b \quad (51)$$

Obtaining the bending solution of a plate using SVRPT is almost same as that involved in case of CPT.

For the case of vibrations of a plate, the differential equation is given by

$$D\nabla^2\nabla^2 w_b - \frac{\rho h^3}{12}\left[1+\frac{12}{5(1-\mu)}\right]\frac{\partial^2}{\partial t^2}\left(\nabla^2 w_b\right) + \rho h\frac{\partial^2 w_b}{\partial t^2} + \frac{\rho^2 h^3(1+\mu)}{5}\frac{\partial^4 w_b}{\partial t^4} = q(x,y,t) \quad (52)$$

The above equation involves the terms pertaining to shear deformation as well as the terms related to rotary inertia.

4. Conclusion

In this paper, a short discussion on classical and refined beam and plate theories has been discussed. An attempt has been made to familiarize the reader about the different class of beam and plate theories such as classical, first-order and higher-order theories. The necessity of shear deformation theories in case of thick beam and plate theories has been highlighted. Throughout the paper, the discussion is restricted to the beam and plate theories suitable for the analysis of plates made of isotropic materials. For the case of laminated plates and plates made of functionally graded materials, the reader is suggested to further explore the beam and plate theory literature. Also, nonlocal beam and plate theories are available for the analysis of micro-scale and nano-scale beams and plates. The literature on applications of beam and plate theories for the case of beams and plates made of smart or intelligent materials is also very interesting.

References

1. Y. M. Ghugal and R. P. Shimpi, “A review of refined shear deformation theories for isotropic and anisotropic laminated beams,” J Reinf Plast Compos, Vol. 20, pp. 255_272, 2001.
2. S. P. Timoshenko, “On the correction for shear of the differential equation for transverse vibrations of prismatic bars,” Philos Mag, Vol. 41, pp. 744_746, 1921.
3. M. Levinson, “A new rectangular beam theory,” J Sound Vib, Vol. 74, pp. 81–87, 1981.
4. P. R. Heyliger and J. N. Reddy, “A Higher Order Beam Element for Bending and Vibration Problems,” J Sound Vib, Vol. 126, pp. 309_326, 1988.
5. Y. M. Ghugal and R P Shimpi, “A review of refined shear deformation theories of isotropic and anisotropic laminated plates,” J. Reinf. Plast. Compos., Vol. 21, pp. 775_813, 2002.
6. E. Reissner, “The effect of transverse shear deformation on the bending of elastic plates,” J. Appl. Mech., Vol. 12, 69_77, 1945.
7. R. D. Mindlin, “Influence of rotary inertia and shear on flexural motions of isotropic, elastic plates” J. Appl. Mech., Vol. 18, pp. 31_38, 1951.
8. M. Levinson, “An accurate, simple theory of the statics and dynamics of elastic plates.” Mech. Res. Commun. Vol. 7, pp. 343_350, 1980.
9. J. N. Reddy, “A simple higher-order theory for laminated omposite plates,” J. Appl. Mech., Vol. 51, pp. 745_752, 1984.
10. R. P. Shimpi, “Refined plate theory and its variants,” AIAA J, Vol. 40, pp. 137_146, 2002.
11. R. P. Shimpi, H. G. Patel and H. Arya, “New first-order shear deformation plate theories.” J. Appl. Mech., Vol. 74, pp. 523_533, 2007.
12. I. H. Shames and C. L. Dym, “Energy and finite element methods in structural mechanics,” Washington: Hemisphere Publishing Corporation, 1985, pp.185–204; 257–266; 323–350.
13. C. M. Wang, J. N. Reddy and K. H. Lee, “Shear deformable beams and plates: Relationships with classical solutions,” Oxford: Elsevier Science, pp. 11_23, 2000.
14. R. P. Shimpi, R. A. Shetty and A. Guha, “A Simple Single Variable Shear Deformation Theory for A Rectangular Beam,” Proc. Inst. Mech. Eng., Part C, Vol. 231, pp. 4576-4591, 2017.
15. R. P. Shimpi, R. A. Shetty and A. Guha, “A Single Variable Refined Theory for Free Vibrations of a Plate Using Inertia Related Terms in Displacements.” Eur. J. Mech. A. Solids., Vol. 65, pp. 136-148, 2016.